

SCHUR-FINITENESS AND ENDOMORPHISMS UNIVERSALLY OF TRACE ZERO VIA CERTAIN TRACE RELATIONS

ALESSIO DEL PADRONE AND CARLO MAZZA

ABSTRACT. We provide a sufficient condition that ensures the nilpotency of endomorphisms universally of trace zero of Schur-finite objects in a category of homological type, i.e., a \mathbb{Q} -linear \otimes -category with a tensor functor to super vector spaces. This generalizes previous results about finite-dimensional objects, in particular by Kimura in the category of motives. We also present some facts which suggest that this might be the best generalization possible of this line of proof. To get the result we prove an identity of trace relations on super vector spaces which has an independent interest in the field of combinatorics. Our main tool is Berele-Regev's theory of Hook Schur functions. We use their generalization of the classic Schur-Weyl duality to the "super" case, together with their factorization formula.

1. INTRODUCTION

Let \mathcal{A} be a \mathbb{Q} -linear tensor category in which idempotents split equipped with a functor H to super vector spaces. The endomorphisms universally of trace zero of an object are the endomorphisms whose compositions with any other endomorphisms have all trace zero. In the case \mathcal{A} is a category of motives, the endomorphisms $\mathcal{N}(A)$ universally of trace zero of an object A are a subset of the numerically trivial ones. According to a result by Kimura in [17], if an object A is finite-dimensional, then every numerically trivial endomorphism of A is nilpotent. A still open question is whether the same result holds for Schur-finite objects (see [8]).

Finiteness conditions for motives are related to part of the standard conjectures: in particular, if \mathcal{A} is the category of Chow motives, then the finiteness of the motive of a surface with $p_g = 0$ is equivalent to Bloch's Conjecture (see [12, Theorem 7]). In this paper we show (Theorem 2.4) that if A has the sign property (Definition 2.3) and $S_\lambda(A) = 0$, where λ is a partition which is not too big, i.e., it does not contain the rectangle with $a+2$ rows and $b+2$ columns, a and b being the dimensions of the even and the odd part of $H(A)$, then every endomorphism in $\mathcal{N}(A)$ is nilpotent.

We start by recalling the definitions for the different finiteness notions and their main properties. We then recall the nilpotency conjecture and how this relates to the various finiteness notions. Then the main result is stated and proved (modulo a combinatorial result) and we dedicate the rest of the section to analyzing some related facts and reasons why this might be the sharpest result possible using this particular line of proof. The second and last section of the paper will prove the necessary combinatorial facts which are used in the proof of the main result.

We keep the notation and the terminology of [8], except that we will write Tr for the categorical trace in the sense of [15], tr for the ordinary trace of matrices, and

we will write $V_0|V_1$ ($d_0|d_1$) for (the dimension of) the super vector space having the d_i -dimensional vector space V_i in degree i .

Since neither of the authors is an expert in combinatorics, the third section will be less concise and we now recall some basic facts and notations: a partition λ of n is a sequence of integers $(\lambda_1, \dots, \lambda_r)$ such that $\lambda_i \geq \lambda_{i+1} > 0$ for all $i = 1, \dots, r-1$ and $\sum_i \lambda_i = n$. We will often confuse a partition with its associated Young diagram and, e.g., we say that the partition (b^a) is the rectangle with a rows and b columns. If λ is a partition, we write $\lambda' = (\lambda'_1, \dots, \lambda'_s)$ for the transposed partition. We say that $(i, j) \in \lambda$ if $\lambda_i \geq j$. The maximal hook of λ is the hook $(\lambda_1, 1^{r-1})$; the maximal skew hook is the set $\{(i, j) \in \lambda \text{ s.t. } (i+1, j+1) \notin \lambda\}$. If ν is the maximal (skew or not) hook, then $\lambda \setminus \nu$ is the partition $(\lambda_2 - 1, \dots, \lambda_r - 1)$. Let $\mu = (\mu_1, \dots, \mu_s)$ and $\lambda = (\lambda_1, \dots, \lambda_r)$ be two partitions, we say that $\mu \subseteq \lambda$ if $s \leq r$ and $\mu_i \leq \lambda_i$ for all $i = 1, \dots, s$.

2. FINITE-DIMENSIONALITY AND NILPOTENCY

Let \mathcal{A} be a **pseudo-abelian \otimes -category**, i.e., a “ \otimes -catégorie rigide sur F ” as in [1, 2.2.2] in which idempotents split. We assume that $F = \text{End}_{\mathcal{A}}(\mathbb{1})$ and it contains \mathbb{Q} . The partitions λ of an integer n give a complete set of mutually orthogonal central idempotents

$$d_\lambda := \frac{\dim V_\lambda}{n!} \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \sigma$$

in the group algebra $\mathbb{Q}\Sigma_n$ (see [11]). We define an endofunctor on \mathcal{A} by setting $S_\lambda(A) = d_\lambda(A^{\otimes n})$. This is a multiple of the classical Schur functor corresponding to λ . In particular, we define $\text{Sym}^n(A) = S_{(n)}(A)$ and $\Lambda^n(A) = S_{(1^n)}(A)$. The following definitions are directly inspired by [9] and [17] (see [3], [12], and [19] for further reference).

Definition 2.1. An object A of \mathcal{A} is **Schur-finite** if there is a partition λ such that $S_\lambda(A) = 0$. If $S_\lambda(A) = 0$ with λ of the form (n) (respectively, $\lambda = (1^n)$) then A is called **odd** (respectively, **even**). We say that A is **finite-dimensional** (in the sense of Kimura-O’Sullivan) if $A = A_+ \oplus A_-$ with A_+ even and A_- odd.

Both finite-dimensionality and Schur-finiteness are stable under direct sums, tensor products, duals, and taking direct summands. Hence every finite-dimensional object is Schur-finite, but the converse does not hold (see [6, 2.6.5.1]). On the other hand, this weaker condition is compatible with triangulated structures on the category while finite-dimensionality is not (see [19, 3.6 and 3.8]).

One of the most important consequences of finite-dimensionality is the nilpotency of endomorphisms universally of trace zero.

Definition 2.2. Recall that we have F -linear **trace** maps $\text{Tr}: \text{End}_{\mathcal{A}}(A) \longrightarrow \text{End}_{\mathcal{A}}(\mathbb{1})$ compatible with \otimes -functors. We define the F -submodules of **endomorphisms universally of trace zero** as

$$\mathcal{N}(A) := \{f \in \text{Hom}_{\mathcal{A}}(A, A) \mid \text{Tr}(f \circ g) = 0, \text{ for all } g \in \text{Hom}_{\mathcal{A}}(A, A)\}.$$

We say an object A is a **phantom** if $\text{Id}_A \in \mathcal{N}(A)$.

André and Kahn proved in [3, 9.1.14] that if A is a finite-dimensional object, then any $f \in \mathcal{N}(A)$ is nilpotent. In particular, if all objects of \mathcal{A} are finite-dimensional, then the projection functor $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{N}$ lifts idempotents and is conservative (hence

“there are no phantom objects”). In general \otimes -categories, Schur-finiteness is not sufficient to get the nilpotency of $\mathcal{N}(A)$; see [3, 10.1.1] for an example of a phantom non-zero Schur-finite object, i.e., whose identity is universally of trace zero.

In the special case of Chow motives, if $M(X)$ is the motive of a smooth variety X , then $\mathcal{N}(M(X))$ is the set of the numerically trivial correspondences $CH^{\dim X}(X \times X)_{\text{num}}$, and the André-Kahn result generalizes a previous result by Kimura ([17, 7.5]) who also conjectured in *loc. cit.* that all Chow motives are finite-dimensional, and hence all $\mathcal{N}(M(X))$ are nilpotent. Moreover, the conjectures of Bloch-Beilinson-Murre (together with the “numerical=homological” standard conjecture) imply the nilpotency of all endomorphism algebras and this implies the finite-dimensionality of each object. In order to extend the André-Kahn result to a larger subclass of Schur-finite objects, we needed to find a peculiar feature which forces the nilpotency and is expected to be true in the category of motives. We will show that the sign property is such a feature.

From now on, let \mathcal{A} be a category of **homological type** (see [16, 4]), i.e., a category with a \otimes -functor to super vector spaces $H : \mathcal{A} \rightarrow sVect$ which we will call “cohomology” by abuse of notation.

Definition 2.3. We say that an object A in a category of homological type has the **sign property** if the projections on the even and the odd part of the cohomology $H(A) = H(A)_0 \oplus H(A)_1$ lift to endomorphisms in \mathcal{A} (cf. [16, 4.8]).

In particular, the category of motives is of homological type and the sign property is known as the sign conjecture ([14, p. 426]) which is a part of the conjecture on the algebraicity of the Chow-Künneth decomposition of the diagonal. The main difference is that we do not require the lifts to be idempotents or orthogonal, although they are so in cohomology.

The next theorem is our main result: its proof relies on a combinatorial result which will be proved in §3. The rest of this section will be dedicated to some related remarks some of which suggest that this might be the best generalization possible of this line of proof.

Theorem 2.4. *Suppose A has the sign property and let $H(A)$ be of dimension $d_0|d_1$. Let $S_\lambda(A) = 0$ for a partition λ of $n \geq 2$ such that $(d_1 + 2)^{(d_0+2)} \not\subseteq \lambda$ and let s be the length of the biggest hook in λ . Then for any $f \in \mathcal{N}(A)$ we have $f^{\circ(s-1)} = 0$.*

Proof. Let ν be the maximal hook ν of λ , $r := n - s$ and $\delta := \lambda \setminus \nu$. A key role is played by the following function: for any $f_1, \dots, f_r \in \text{End}(A)$, let

$$(2.4.1) \quad y(\delta; f_1, \dots, f_r) := \text{Tr}(\mathbf{d}_\delta \circ f_1 \otimes \dots \otimes f_r) \\ = \frac{\dim V_\delta}{|\delta|!} \sum_{\sigma \in \Sigma_r} \chi_\delta(\sigma) \prod_{j=1}^q \text{Tr}(f_{\gamma_j^{l_j-1}(k_j)} \circ \dots \circ f_{\gamma_j(k_j)} \circ f_{k_j}),$$

where $\gamma_1 \circ \dots \circ \gamma_{q_\sigma}$ is the cycle decomposition of σ , k_j is any element in the support of γ_j , and l_j is the length of the cycle γ_j . Our proof of [8, Theorem 2.1] shows that if $y(\delta; -, \dots, -)$ is not the zero function on $\text{End}(A)^r$, then $f^{\circ(s-1)} = 0$ for each $f \in \mathcal{N}(A)$.

Since A has the sign property then there exist two endomorphisms π_0 and π_1 such that $t_i := \text{Tr}(H(\pi_i)) = (-1)^i d_i$ ($i = 0, 1$), and we have the following trace identities:

- (**TI1**) $t_i = \text{Tr}(\pi_i) = \text{Tr}(\pi_i^l) \in \mathbb{Z}$ for all $l > 0$, and
 (**TI2**) $\text{Tr}(\pi_{j_1} \circ \dots \circ \pi_{j_k}) = 0$ for any $k > 1$ and any non-constant map $j: \{1, \dots, k\} \rightarrow \{0, 1\}$.

Let us choose $f_1 = \dots = f_r = g := \alpha_0 \pi_0 + \alpha_1 \pi_1$, then it is an immediate consequence of the properties (**TI1**) and (**TI2**) that y becomes a polynomial in $\alpha_0, \alpha_1, t_0, t_1$

$$y(\delta; g) := y(\delta; g, \dots, g) = \frac{\dim V_\delta}{|\delta|!} \sum_{\sigma \in \Sigma_{|\delta|}} \chi_\delta(\sigma) \prod_{j=1}^q (\alpha_0^{l_j} t_0 + \alpha_1^{l_j} t_1)$$

Since H is a tensor functor, $S_\lambda(H(A)) = H(S_\lambda(A))$. But $S_\lambda(H(A)) = 0$ if and only if $(d_0 + 1, d_1 + 1) \in \lambda$ (see [9, 1.9]) and therefore $S_\lambda(A) = 0$ implies that $(d_0 + 1, d_1 + 1) \in \lambda$, and, in particular, $(d_0, d_1) \in \delta$. But by hypothesis, we have that $(d_0 + 2, d_1 + 2) \notin \lambda$ and then $(d_0 + 1, d_1 + 1) \notin \delta$. So (d_0, d_1) is in the maximal skew hook of δ .

From the results in §3, it follows that $y(\delta; g)$ is the polynomial $P(\delta; \alpha_0, \alpha_1; t_0, t_1)$ in $\mathbb{Q}[\alpha_0, \alpha_1, t_0, t_1]$ which, when computed for $t_0 = d_0$ and $t_1 = -d_1$, is a non-zero polynomial in α_0 and α_1 . Since the coefficients of this polynomial are a field of characteristic zero, this proves the theorem. \square

Remark 2.5. Proving that the function $y(\lambda \setminus \nu; -, \dots, -)$ (defined in equation 2.4.1) is not zero is a combinatorial problem because it does not depend on the choice of the category. In particular, it can be calculated on a super-vector space.

Remark 2.6. If $S_\lambda(A) = 0$ and its cohomology is of super dimension $d_0|d_1$, then $\lambda \supseteq ((d_1 + 1)^{d_0+1})$, and there exist f_1, \dots, f_r such that $y(\lambda \setminus \nu; f_1, \dots, f_r) \neq 0$ only if $\lambda \not\supseteq ((d_1 + 2)^{d_0+2})$.

The first assertion is in the proof of 2.4, while the second one follows from the universal relations among super traces characterized by Razmyslov. A suitable formulation of his result is in Berele's [4, 3.1] where it is shown that if $(d_0 + 2, d_1 + 2) \in \lambda$, and so $(d_0 + 1, d_1 + 1) \in \delta := \lambda \setminus \nu$, then $\mathbf{d}_\delta \in \mathbb{Q}\Sigma_{|\delta|}$ belongs to the two sided ideal of $\mathbb{Q}\Sigma_{|\delta|}$ of those elements whose associated “trace polynomial” (in our notation, just a multiple of $y(\delta; -, \dots, -)$ by a non-zero constant) is an identity of super matrices with even dimension d_0 and odd dimension d_1 . That is, if $\lambda \supseteq ((d_1 + 2)^{d_0+2})$, then $y(\lambda \setminus \nu; f_1, \dots, f_r) = 0$ for any choice of f_1, \dots, f_r . For more details, see the given reference, or [7], or [6, 2.5].

Remark 2.7. Note that in categories of homological type, and in particular for motives, there is really nothing lost in considering only $\text{End}_{\mathcal{A}}(A)^r$ instead of trying to exploit $\text{End}_{\mathcal{A}}(A^{\otimes r})$, for $\text{End}(H(A^{\otimes r})) \cong \text{End}(H(A))^{\otimes r}$ as vector spaces.

Remark 2.8. (P. Deligne) By 2.5, it suffices to do the calculations on a super vector space V . Letting π_0 and π_1 be the projections, $\pi_0 - \pi_1$ is $(-1)^i$ in degree i and so

$$\text{Tr}(\mathbf{d}_\delta \circ (\pi_0 - \pi_1)^{\otimes r}) = \dim(\text{Im}(\mathbf{d}_\delta)^+) + \dim(\text{Im}(\mathbf{d}_\delta)^-)$$

which is non-zero if and only if $S_\delta(V) \neq 0$. This is a sufficient condition for the non-vanishing of $y(\delta; \pi_0 - \pi_1)$.

Remark 2.9. By 2.4, we have that $f^{\circ(s-1)} = 0$ for all $f \in \mathcal{N}(A)$. Using Razmyslov's improvement of the (Dubnov-Ivanov) Nagata-Higman Theorem (see [10] and [20, 11.8.10]) we have that $\mathcal{N}(A)^{(s-1)^2} = 0$. Thus for finite-dimensional objects $A =$

$A_+ \oplus A_-$ the known nilpotency bounds ([2, 3.4]) can be improved: indeed such an object is (minimally) killed by the rectangle $\lambda = ((\text{kim}(A_-) + 1)^{\text{kim}(A_+) + 1})$, hence $s - 1 = \text{kim}(A)$ ([6, 2.4.10] or [7]). In the setting of categories of homological type a different bound is given in [16, 4.10 b)] (see also [16, 4.11]). However, for any (twist of direct summand of a) pure motive $\mathfrak{h}(X)$, the global nilpotency bound for $\mathcal{N}(\mathfrak{h}(X))$ should be $\dim(X) + 1$ by Bloch-Beilinson-Murre's conjectures ([13, Conjecture 2.1 (strong e)]). As a partial evidence, Morihiko Saito proved in [21] that for smooth complex surfaces S with $p_g = 0$, Bloch's conjecture ([13, Conjecture 1.8]) is equivalent to $(CH^2(S \times S)_{\text{hom}})^3 = 0$.

3. THE COMBINATORIAL RESULT

We now state and prove the main combinatorial result. As a corollary, we will get the non-vanishing we needed. We are interested in studying the following polynomial.

Definition 3.1. Let δ be a partition of r , we define:

$$P(\delta; \alpha_0, \alpha_1; t_0, t_1) := \frac{\dim V_\delta}{r!} \sum_{\sigma \in \Sigma_r} \chi_\delta(\sigma) \prod_{j=1}^q (\alpha_0^{l_j} t_0 + \alpha_1^{l_j} t_1)$$

as a polynomial in $\mathbb{Q}[\alpha_0, \alpha_1, t_0, t_1]$, where the l_j are the lengths of the cycles in the cycle decomposition of σ .

Notice that $P(\delta; 1, 0; t_0, t_1)$ is the content polynomial of δ as an element of $\mathbb{Q}[t_0]$.

Proposition 3.2. *Let δ be any partition, and let $P(\delta; \alpha_0, \alpha_1; t_0, t_1)$ be the polynomial defined in 3.1. Then for any (d_0, d_1) in the maximal skew hook of δ we have the following identity of polynomials in α_0 and α_1 :*

$$\begin{aligned} & P(\delta; \alpha_0, \alpha_1; d_0, -d_1) \\ &= (\dim V_\delta)(-1)^{|\nu|} \frac{\dim V_\mu}{|\mu|!} \frac{\dim V_\nu}{|\nu|!} (\alpha_0 - \alpha_1)^{d_0 d_1} \alpha_0^{|\mu|} \alpha_1^{|\nu|} \text{cp}_\mu(d_0) \text{cp}_\nu(d_1), \end{aligned}$$

where μ is the partition whose non-zero parts are the positive $\delta_i - d_1$, and ν is the partition whose non-zero parts are the positive $\delta'_i - d_0$ (see [5, 6.14]).

Proof. Let \mathcal{A} be the \mathbb{Q} -linear category of super vector spaces and let A be $\mathbb{Q}^{d_0} | \mathbb{Q}^{d_1}$. By Remark 2.5, we can describe the polynomial $P(\delta; \alpha_0, \alpha_1; d_0, -d_1)$ in terms of the (super) trace of $g \in \text{End}_{\mathcal{A}}(A)$ given by $\alpha_0 \pi_0 + \alpha_1 \pi_1$, where $\pi_i \in \text{End}_{\mathcal{A}}(A)$ is the projection on the degree i part (if $\alpha_0 \alpha_1 \neq 0$ then $g \in GL(d_0 | d_1)$). More precisely, if e_δ is any (idempotent) Young symmetrizer associated to δ , then $\overline{P} = \frac{1}{\dim V_\delta} P(\delta; \alpha_0, \alpha_1; d_0, -d_1)$ is the evaluation on $\tilde{g} := \alpha_0 \pi_0 - \alpha_1 \pi_1$ of the character χ_δ of the representation $e_\delta^A A^{\otimes |\delta|}$ of $GL(d_0 | d_1)$. Indeed, letting s_θ be the usual Schur

function of a partition ϑ we have:

$$\begin{aligned}
\overline{P}(\delta; \alpha_0, \alpha_1; d_0, -d_1) &= \frac{1}{\dim V_\delta} \text{Tr}(d_\delta \circ g^{\otimes |\delta|}) \\
&= \text{Tr}(e_\delta \circ g^{\otimes |\delta|}) \\
&= \text{tr}(e_\delta \circ \widehat{g}^{\otimes |\delta|}) \\
&= \text{tr}(e_\delta \circ (\alpha_0 \pi_0 - \alpha_1 \pi_1)^{\otimes |\delta|}) \\
&\stackrel{(a)}{=} \text{HS}_\delta(\underbrace{\alpha_0, \dots, \alpha_0}_{d_0}, \underbrace{-\alpha_1, \dots, -\alpha_1}_{d_1}) \\
&\stackrel{(b)}{=} \left(\prod_{i=1}^{d_0} \prod_{j=1}^{d_1} (\alpha_0 - \alpha_1) \right) s_\mu(\underbrace{\alpha_0, \dots, \alpha_0}_{d_0}) s_\nu(\underbrace{-\alpha_1, \dots, -\alpha_1}_{d_1}) \\
&= (-1)^{|\nu|} (\alpha_0 - \alpha_1)^{d_0 d_1} \alpha_0^{|\mu|} \alpha_1^{|\nu|} s_\mu(\underbrace{1, \dots, 1}_{d_0}) s_\nu(\underbrace{1, \dots, 1}_{d_1}), \\
&\stackrel{(c)}{=} (-1)^{|\nu|} \frac{\dim V_\mu}{|\mu|!} \frac{\dim V_\nu}{|\nu|!} (\alpha_0 - \alpha_1)^{d_0 d_1} \alpha_0^{|\mu|} \alpha_1^{|\nu|} \text{cp}_\mu(d_0) \text{cp}_\nu(d_1),
\end{aligned}$$

where μ is the partition whose non-zero parts are the positive $\delta_i - d_1$, and ν is the partition whose non-zero parts are the positive $\delta'_i - d_0$ (see [5, 6.14]).

The hypothesis $(d_0 + 1, d_1 + 1) \notin \delta$ allows us to exploit [5, 6.10 (b)] in (a), and [5, 6.20] in (b). Notice that the $\Sigma_{|\delta|}$ action defined in [5, 1] coincides with the one induced by the (Koszul) commutativity constraint of \mathcal{A} . The equality (c) is a well-known fact (see [18, I.3, Example 5 and the proof of I.7(7.6)]). \square

Corollary 3.3. *Let P be the polynomial defined in 3.1. If (d_0, d_1) is in the maximal skew hook of δ , then $P(\delta; \alpha_0, \alpha_1; d_0, -d_1)$ is a non-zero polynomial in α_0 and α_1 .*

Proof. By Theorem 3.2, we just need to show that $\text{cp}_\mu(d_0) \neq 0$ and $\text{cp}_\nu(d_1) \neq 0$. Let us show the first one, since the second one comes from passing to the transpose partition. If (d_0, d_1) is in the maximal skew hook of δ , then $\delta_{d_0+1} \leq d_1$, so μ has at most d_0 rows which in turn implies that $\text{cp}_\mu(d_0) \neq 0$. \square

Shortly after the proof of the Theorem 3.2 was completed, Christine Bessenrodt informed us that Richard Stanley had suggested a slightly different solution: it follows from the theory of super Schur functions, for our polynomial P can be obtained from the power-sum expansion of the Schur function s_δ by a suitable substitution. Then, one can apply Problem 50(g) in the Supplementary Problems for Chapter 7 of Stanley's book Enumerative Combinatorics II (on his EC web page), which is based on [5]. We decided to maintain our approach for it is closer to our original "motivic" point of view.

ACKNOWLEDGMENTS

We thank Christine Bessenrodt for her suggestions and interest in the problem, and Bruno Kahn for supplying the most recent version of his work. The first author wishes also to express his gratitude to Uwe Jannsen for his constant support and lots of useful conversations during a postdoc stay at the math department of the University of Regensburg.

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E-mail address: `delpadro@dima.unige.it`

E-mail address: `mazza@dima.unige.it`